

# Bijjective combinatorics of positive braids

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## Abstract

We give a new and bijective proof for the formula of the growth function of the positive braid monoid with respect to Artin generators.

## Introduction

Consider  $n + 1$  strands numbered from 1 to  $n + 1$  and  $n$  elementary moves  $\sigma_1, \dots, \sigma_n$  where  $\sigma_i$  represents the crossing of strand  $i$  and  $i + 1$  with strand  $i$  above. Starting from an uncrossed configuration we apply sequences of elementary moves to obtain a braided configuration; two configurations are equivalent if one can be obtained from the other only by moving the strands without touching the top and bottom extremities. An equivalence class of configuration is called a *braid*. The set of braids, with concatenation of two braids as internal law, is a monoid. It is precisely the positive braid monoid on  $n + 1$  strands, denoted  $\mathbb{P}$ , generated by  $\Sigma = \{\sigma_1, \dots, \sigma_n\}$  (called *Artin generators*) and subject to the relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2 \quad (0.1)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 1 \quad (0.2)$$

For  $u \in \mathbb{P}$  there is a natural length function defined by:

$$|u|_\Sigma = \min\{k \mid \exists u_1, \dots, u_k \in \Sigma \text{ such that } u = u_1 \dots u_k\}$$

The growth function of the positive braid monoid for the Artin generators is defined by:

$$F(t) = \sum_{b \in \mathbb{P}} t^{|b|_\Sigma}$$

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Different ideas have been used to compute the growth function of the positive braid monoid. In [Bra91], Brazil use the fact that, to each braid is associated a unique decomposition usually called its normal form (or its Garside's normal form or Thurston's normal form). This set of normal forms is regular, meaning that it is recognized by a finite state automaton. From this automaton's adjacency matrix, we can obtain directly the growth function. But it is not a very efficient way to compute it, as for braids on  $n$  strands, this automaton has  $n!$  states and then it takes a time exponential in the number of strands to get the formula. In [Deh07], Dehornoy gives a method to reduce from  $n!$  to  $p(n)$  (where  $p(n)$  is the number of partitions of  $n$ ) the number of states of this automaton.

Bronfman (see [Bro01]) and Krammer (see chapter 17 of [Kra05]) give a new method to compute the growth function in quadratic time. Their proof is based on an inclusion-exclusion principle. We give here a different and bijective proof of this result.

To explain our point of view, let look at the history of results for trace monoids. Trace monoids (also called “heaps of pieces monoids” or “free partially commutative monoids”) denoted  $\mathbb{M}$  are defined by the following semi-group presentation:

$$\mathbb{M} = \langle \Sigma \mid ab = ba \text{ if } (a, b) \in I \rangle,$$

where  $\Sigma$  is a finite set of generators and  $I$  is a symmetric and antireflexive relation of  $\Sigma \times \Sigma$  called the commutation relation. In 1969, Cartier and Foata computed the growth function of these monoids by using an inclusion-exclusion principle to get a Möbius inversion formula (see [CF69]). The proofs of Bronfman and Krammer use the same kind of arguments.

In [Vie86], Viennot gives a new way to compute the growth function of heaps of pieces monoids. To perform this, he considers the set  $\mathcal{G}$  of all heaps of pieces of height at most one and constructs an involution from  $\mathcal{G} \times \mathbb{M}$  into itself. This involution matches elements of monoids two by two and makes it easy to count them. However, the construction of this pairing provides more than a new way to compute the growth function. It gives indeed an additional combinatorial understanding of trace monoids. Several byproducts are given in [Vie86].

In the present paper, we show that Viennot's proof can be extended to braid monoids, which gives a new point of view in the combinatorics of braids. More precisely, we explain in 1.2 how to define a set  $\mathcal{G}$  of simple braids and how to use it to construct an involution from  $(\mathcal{G} \times \mathbb{P})$  to itself.

Following Bronfman, we show that Viennot's idea can actually be extended to a wider class of monoids which naturally includes braid monoids

and trace monoids but also Artin-Tits monoids and Birman-Ko-Lee braid monoids among others.

# 1 Growth function of braid monoids

## 1.1 Presentation of braid monoids

We denote by  $\Sigma$  the set  $\{\sigma_1, \dots, \sigma_n\}$  and by  $\Sigma^*$  the free monoid on  $\Sigma$ . That is to say  $\Sigma^*$  is the set of finite words on the alphabet  $\Sigma$  with concatenation as monoid law. We denote by 1 the empty word.

The *positive braid monoid*  $\mathbb{P}$  on  $n+1$  strands has the following semigroup presentation:

$$\mathbb{P} = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \leq 2 \rangle$$

We denote by  $\prec$  divisibility on the left in  $\mathbb{P}$  (that is, for  $a, b \in \mathbb{P}$ , we have  $a \prec b$  if and only if there exists  $c \in \mathbb{P}$  such that  $ac = b$ ) and denote similarly by  $\succ$  divisibility on the right.

For  $u \in \mathbb{P}$ , set

$$\text{left}(u) = \{\sigma \in S \text{ such that } \sigma \prec u\}$$

$$\text{right}(u) = \{\sigma \in S \text{ such that } u \succ \sigma\}$$

## 1.2 A few notations

We define a new set  $\mathcal{G}$  of generators of  $\mathbb{P}$ . For all  $j, i \in \{1, \dots, n\}$  such that  $j + i \leq n + 1$ , set:

$$\delta_{\{j, j+1, \dots, j+i\}} = (\sigma_{j+i-1}) \dots (\sigma_{j+1})(\sigma_j)$$

The element  $\delta_{\{j, \dots, j+i\}}$  is the braid where strand  $j + i$  moves up to position  $j$  “behind the braid”. We set then:

$$\Delta_{\{j, j+1, \dots, j+i\}} = \delta_{\{j, j+1\}} \cdot \delta_{\{j, j+1, j+2\}} \cdot \dots \cdot \delta_{\{j, \dots, j+i\}} \quad (1.1)$$

$$= (\sigma_j)(\sigma_{j+1}\sigma_j) \dots (\sigma_{j+i-1} \dots \sigma_{j+1}\sigma_j), \quad (1.2)$$

The element  $\Delta_{\{j, \dots, j+i\}}$  is the half-turn of the strands  $j, j + 1, \dots, j + i$  (see Fig.1.2 for examples).

Set  $\mathcal{G}$  to be:

$$\mathcal{G} = \bigcup_{J_1 \cup \dots \cup J_p} (\Delta_{J_1} \dots \Delta_{J_p}) \quad (1.3)$$

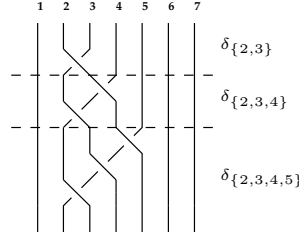


Figure 1: Decomposition in  $\mathbb{P}_7$  of  $\Delta_{\{2,3,4,5\}}$  as a product of  $\delta_{\{2,3\}}$ ,  $\delta_{\{2,3,4\}}$  and  $\delta_{\{2,3,4,5\}}$ .

where  $J_1, \dots, J_p$  are disjoint subsets formed by consecutive integers of  $\{1, \dots, n+1\}$ , such that  $J_l < J_{l+1}$  for all  $l \in \{1, \dots, p-1\}$  (i.e.: if  $i_l \in J_l$  and  $i_{l+1} \in J_{l+1}$  then  $i_l < i_{l+1}$ ) and such that  $|J_l| \geq 2$  for all  $l \in \{1, \dots, p\}$ .

We use the convention that if  $J$  is a singleton,  $\Delta_J$  is the empty word. But in the following, we always write an element of  $G$  in the simplest possible way (i.e.: without terms equal to the empty word).

We introduce a few notations:

- Let  $\tau : \mathcal{G} \rightarrow \mathbb{N}$  be the function defined by :

$$\tau(g) = \sum_{i=1}^p (|J_i| - 1) \text{ if } g = \Delta_{J_1} \cdot \dots \cdot \Delta_{J_p}. \quad (1.4)$$

So  $\tau(g)$  is the number of different Artin generators which appear in a representative of  $g$ .

- Let  $u$  be a word of  $S^*$  such that  $u = v \cdot w$  with  $v, w \in S^*$ , we set:  $v^{-1} \cdot u = w$  and  $u \cdot w^{-1} = v$ .
- Let  $J = \{i, i+1, \dots, i+j\}$  be a set of consecutive integers, we denote:
  1.  $m(J) = i + j - 1$
  2.  $J + 1 = J \cup \{i + j + 1\} = \{i, \dots, i + j, i + j + 1\}$
  3.  $J - 1 = J \setminus \{i + j\} = \{i, \dots, i + j - 1\}$

*Remark 1.* Let  $g = \Delta_{J_1} \dots \Delta_{J_p}$  be an element of  $\mathcal{G}$ . We can notice that  $g$  is the least common multiple of  $\bigcup_{i=1}^p (J_i - 1)$ .

Besides if  $g \in \mathcal{G}$  is the least common multiple of  $\Sigma$ , then  $\tau(g) = |\Sigma|$ .

### 1.3 Growth function for the braid monoids

We can now state the main result:

**Theorem 1.** *In  $\mathbb{Z}\langle\langle \mathbb{P} \rangle\rangle$ , the following identity holds:*

$$\left(\sum_{g \in \mathcal{G}} (-1)^{\tau(g)} g\right) \cdot \left(\sum_{b \in \mathbb{P}} b\right) = 1 \quad (1.5)$$

**Corollary 1.** *The growth function of the positive braid monoid is equal to:*

$$F(t) = \sum_{b \in \mathbb{P}} t^{|b|_S} = \left[ \sum_{g \in \mathcal{G}} (-1)^{\tau(g)} t^{|g|_S} \right]^{-1} \quad (1.6)$$

Proofs of Theorem 1 and Corollary 1 will be given in sections 1.4 and 1.5.

*Example 1* (Explicit formula on 4 strands). On Figure 2, we can read the value of  $\tau$  and the length of all elements of  $\mathcal{G}$  on 4 strands. The growth function is then:

$$F_4(t) = \frac{1}{1 - 3t + t^2 + 2t^3 - t^6}$$

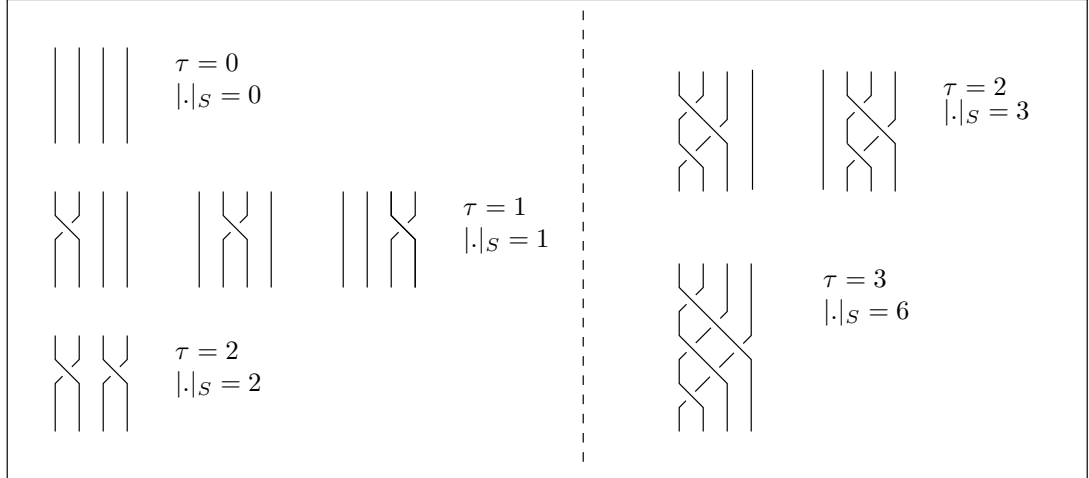


Figure 2: The set  $\mathcal{G}$  on 4 strands

#### 1.4 Definition of the involution of $(\mathcal{G} \times \mathbb{P})$

We construct an involution from  $(\mathcal{G} \times \mathbb{P})$  to itself with only  $(1, 1)$  as fixed point. This gives us a natural way to pair elements of  $(\mathcal{G} \times \mathbb{P}) \setminus (1, 1)$  which will imply great simplifications in the following (see subsection 1.5).

To construct the involution, given a couple  $(g, b) \in (\mathcal{G} \times \mathbb{P})$  we want to move “pieces” of  $g$  (respectively  $b$ ) from  $g$  to  $b$  (respectively from  $b$  to  $g$ ). Let us make it more precise; we define the set  $\mathcal{E}$  of elements which will be allowed to move:

$$\mathcal{E} = \{\delta_J, J \text{ subset of consecutive integers of } \{1, \dots, n+1\} \text{ and } |J| \geq 2\} \quad (1.7)$$

We can now define:

**Definition 1.** Let  $(g, b)$  in  $(\mathcal{G} \times \mathbb{P})$  and  $u \in \mathcal{E}$  such that  $u \prec b$  (resp  $g \succ u$ ). We then set  $g' = g \cdot u$  and  $b' = u^{-1} \cdot b$  (resp.  $g' = g \cdot u^{-1}$  and  $b' = u \cdot b$ ). Now, if  $g'$  satisfies the following conditions:

1.  $g'$  belongs to  $\mathcal{G}$
2.  $\tau(g') = \tau(g) \pm 1$ ,

then  $u$  is called an *eligible moving part* of  $(g, b)$ .

Observe that for  $|J| \geq 2$ ,  $\Delta_J \cdot \delta_J^{-1} = \Delta_{J-1}$ . So there is at least one eligible moving part of  $(g, b)$  if  $g \neq 1$ . For  $(1, b)$  there is clearly at least one eligible moving part unless  $b = 1$ . So for  $(g, b) \neq (1, 1)$ , it exists at least one eligible moving part of  $(g, b)$ . In this case we consider the eligible moving part maximal for the lexicographic ordering induced by the ordering  $\sigma_1 < \sigma_2 < \dots < \sigma_n$  on the generators. We call it the *moving part* of  $(g, b)$  and denote it  $k$ . Since  $k$  cannot be both from  $g$  to  $b$  and from  $b$  to  $g$ , we can set:

$$\Psi(g, b) = \begin{cases} (1, 1) & \text{if } (g, b) = (1, 1) \\ (gk, k^{-1}b) & \text{if } k \text{ is from } b \text{ to } g \\ (gk^{-1}, kb) & \text{if } k \text{ is from } g \text{ to } b \end{cases}$$

Figure 3 shows examples of how  $\Psi$  works on some couples of  $\mathcal{G}_5 \times \mathbb{P}_5$  (eligible moving parts are represented with dashed lines).

**Lemma 1.** The function  $\Psi$  is an involution from  $(\mathcal{G} \times \mathbb{P})$  into itself whose unique fixed point is  $(1, 1)$ .

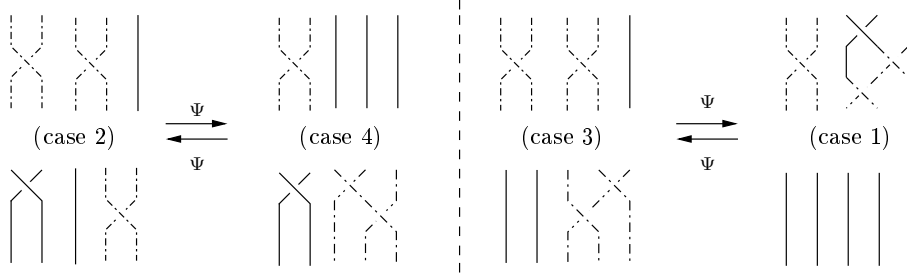


Figure 3: Examples of how  $\Psi$  works. Case numbers refer to lemma 2. Elements of  $\mathcal{G}_5$  are represented at the top and those of  $\mathbb{P}_5$  at the bottom

*Proof.* Assume that  $\Psi$  fixes  $(g, b)$  in  $(\mathcal{G} \times \mathbb{P})$ , it implies that there is no eligible moving part of  $(g, b)$ . Now it implies that  $g = 1$  and then an eligible moving part is any element of  $\text{left}(b)$ , hence  $b = 1$ .

Now, given a couple  $(g, b)$  in  $(\mathcal{G} \times \mathbb{P})$  we look at the maximal eligible moving part. From Definition 1, we can see, by direct inspection, that:

**Lemma 2.** Let  $(g, b)$  in  $(\mathcal{G} \times \mathbb{P})$ . We write  $g = \Delta_{J_1} \cdot \dots \cdot \Delta_{J_p}$ . Let  $\Psi(g, b) = (g', b')$  and let  $\sigma_{\max}$  be the biggest element of  $\text{left}(b)$  then:

1. If  $\sigma_{\max} \leq \sigma_{m(J_p)}$ , then the moving part is  $\delta_{J_p}$  and  $(g', b') = (\Delta_{J_1} \dots \Delta_{J_{p-1}}, \delta_{J_p} \cdot b)$ .
2. If  $\sigma_{\max}$  is equal to  $\sigma_{m(J_p)+1}$  but  $\delta_{J_{p+1}} \not\prec b$ , then the conclusion of the previous case still holds.
3. If  $\sigma_{\max}$  is equal to  $\sigma_{m(J_p)+1}$  and  $\delta_{J_{p+1}} \prec b$ , then the moving part is  $\delta_{J_{p+1}}$ , and  $(g', b') = (\Delta_{J_1} \dots \Delta_{J_{p+1}}, (\delta_{J_{p+1}})^{-1} \cdot b)$ .
4. If  $\sigma_{\max} > \sigma_{m(J_p)+1}$ , then the moving part is  $\sigma_{\max}$  and  $(g', b') = (g \cdot \Delta_{\{\sigma_{\max}, \sigma_{\max+1}\}}, (\Delta_{\{\sigma_{\max}, \sigma_{\max+1}\}})^{-1} \cdot b)$ .

Examples of the different cases of Lemma 2 are given in Figure 3.

We continue the proof of lemma 1. Let  $(g, b)$  be an element of  $(\mathcal{G} \times \mathbb{P})$  and  $(g', b')$  be its image by  $\Psi$ , we want to prove that  $\Psi((g', b')) = (g, b)$ . Two cases have to be considered: either the moving part was from  $g$  to  $b$  (cases 1 and 2 of lemma 2) or the moving part was from  $b$  to  $g$  (cases 3 and 4 of lemma 2).

The former case is far easier. Keeping previous notations:

- if  $|J_p| = 2$ , we get  $g' = \Delta_{J_1} \dots \Delta_{J_{p-1}}$  and  $b' = \delta_{J_p} \cdot b$ . Applying then case 4 of lemma 2 to  $(g', b')$  gives  $\Psi(g', b') = (g, b)$ .
- if  $|J_p| > 2$ , we get  $g' = \Delta_{J_1} \dots \Delta_{J_{p-1}}$  and  $b' = \delta_{J_p} \cdot b$ . Applying then case 3 of lemma 2 to  $(g', b')$  gives  $\Psi(g', b') = (g, b)$ .

We have to be a little more careful for the second case as we have to study how  $\text{left}(b')$  behaves. Let  $k$  be the moving part of  $(g, b)$  from  $b$  to  $g$ , we write  $k = \delta_{\{i, \dots, i+p\}}$ . If  $\sigma$  belongs to  $\text{left}(b')$  then  $\sigma$  is not bigger than  $\sigma_{i+p}$ . Indeed if we assume that there exists  $j > i+p$  such that  $\sigma_j \in \text{left}(b')$ , using the commutation relations, we can rewrite  $b = k \cdot \sigma_j \dots = \sigma_j \cdot k \dots$ . Applying now lemma 2 contradicts the fact that  $k$  is the moving part from  $b$  to  $g$ .

So we are in case 2 or 3. It remains to prove that we are not in case 3 or equivalently that  $\delta_{\{i, \dots, i+p, i+p+1\}} \not\prec b'$ . Assume that  $\delta_{\{i, \dots, i+p, i+p+1\}} \prec b'$  then we can rewrite :

$$\begin{aligned} b &= \delta_{\{i, \dots, i+p\}} \cdot \delta_{\{i, \dots, i+p, i+p+1\}} \cdot b_1 \\ &= \delta_{\{i, \dots, i+p+1\}} \cdot \delta_{\{i+1, \dots, i+p\}} \cdot b_1 \end{aligned}$$

Since  $\delta_{\{i, \dots, i+p+1\}} = \sigma_{i+p} \dots \sigma_i$ , it shows that  $\sigma_{i+p}$  belongs to  $\text{left}(b)$  and once more lemma 2 leads to a contradiction with  $k$  being the moving part from  $b$  to  $g$ . This concludes the proof.  $\square$

## 1.5 Proof of Theorem 1 and Corollary 1

For  $(g, b) \in (\mathcal{G} \times \mathbb{P}) \setminus \{(1, 1)\}$ , let  $\Psi(g, b) = (g', b')$ . We see that  $gb$  and  $g'b'$  are in  $\mathbb{P}$  equal and that  $\tau(g)$  differs from  $\tau(g')$  only by 1. We can translate these two observations into the following equality (in  $\mathbb{Z}\langle\langle \mathbb{P} \rangle\rangle$ ):

$$(-1)^{\tau(g)} gb + (-1)^{\tau(g')} g'b' = 0$$

Besides, according to lemma 1,  $\Psi$  is an involution whose unique fixed point is  $(1, 1)$ . We decompose the sum below on  $(\mathcal{G} \times \mathbb{P})$  along the orbits of  $\Psi$  and deduce the following equalities (in  $\mathbb{Z}\langle\langle \mathbb{P} \rangle\rangle$ ):

$$\left( \sum_{g \in \mathcal{G}} (-1)^{\tau(g)} g \right) \cdot \left( \sum_{b \in \mathbb{P}} b \right) = 1 + \sum_{(g, b) \in (\mathcal{G} \times \mathbb{P}) \setminus \{(1, 1)\}} (-1)^{\tau(g)} gb = 1 \quad (1.8)$$

This concludes the proof of the Theorem 1.



*Proof of Corollary 1.* We begin with the second part of Equality 1.8:

$$1 + \sum_{(g,b) \in (\mathcal{G} \times \mathbb{P}) \setminus \{(1,1)\}} (-1)^{\tau(g)} gb = 1$$

Projecting it into  $\mathbb{Z}[t]$  by identifying  $\sigma_1, \dots, \sigma_n$  with  $t$  and noticing that  $|gb|_S = |g|_S + |b|_S$  give:

$$\begin{aligned} 1 + \sum_{(g,b) \in (\mathcal{G} \times \mathbb{P}) \setminus \{(1,1)\}} (-1)^{\tau(g)} t^{|gb|_S} &= 1 \\ 1 + \sum_{(g,b) \in (\mathcal{G} \times \mathbb{P}) \setminus \{(1,1)\}} (-1)^{\tau(g)} t^{|g|_S} t^{|b|_S} &= 1 \\ \left( \sum_{g \in \mathcal{G}} (-1)^{\tau(g)} t^{|g|_S} \right) \cdot \left( \sum_{b \in \mathbb{P}} t^{|b|_S} \right) &= 1 \end{aligned}$$

This concludes the proof of the corollary.  $\square$

Now that we define an involution for braid monoids, we show that our construction remains valid for a larger class of monoids in the next section.

## 2 Generalisation

### 2.1 Definition of a larger class of monoids

We follow the work of Bronfman (see [Bro01]) to extend our result to a larger class of monoids. From now on we only consider monoids  $\mathbb{M} = \langle \mathcal{S} | \mathcal{R} \rangle$  (where  $\mathcal{S}$  is a finite generating set in  $\mathcal{R}$  a finite set of relations), satisfying the following properties:

1.  $\mathbb{M}$  is homogeneous (ie: all relations in  $\mathcal{R}$  are length-preserving)
2.  $\mathbb{M}$  is left-cancellative, ie: if  $a, u, v \in \mathbb{M}$  are such that  $au = av$  then  $u = v$ .
3. If a subset  $\{s_j | j \in J\}$  of the generating set  $\mathcal{S}$  has a common multiple, then this subset has a least common multiple.

We can notice that Property 1 implies the unicity of a least common multiple if it exists.

Let  $J \subset \mathcal{S}$  such that  $J$  has a common multiple and hence a unique least common multiple by properties 1 and 3. We denote this least common

multiple by  $\bigvee(J)$ . Subsection 1.2 leads us to define a new set  $\mathcal{G}$  of generators as:

$$\mathcal{G} = \{\bigvee(J), \text{ for all } J \subset \mathcal{S} \text{ such that } J \text{ has a common multiple} \}$$

Observe that for  $K \subset J \subset \mathcal{S}$  such that  $J$  has a common multiple,  $K$  has also a common multiple and  $\bigvee(K) \prec \bigvee(J)$  (we recall that  $\prec$  represents the divisibility on the left).

*Notation.* For  $J$  and  $K$  defined as above, we denote:

$$\bigvee(K) \cdot \delta_K^{J \setminus K} = \bigvee(J)$$

For  $g = \bigvee(\{s_{i_1}, \dots, s_{i_p}\}) \in \mathcal{G}$ , we set  $\tau(g) = p$ . This extends the definition of  $\tau$  of subsection 1.2.

## 2.2 Growth function for monoids

**Theorem 2.** *Let  $\mathbb{M}$  be a monoid satisfying Properties 1,2 and 3 of section 2.1 and  $\mathcal{S} = \{s_1, \dots, s_n\}$  be the set of generators of  $\mathbb{M}$ . In  $\mathbb{Z}\langle\langle\mathbb{M}\rangle\rangle$  the following identity holds:*

$$\left(\sum_{g \in \mathcal{G}} (-1)^{\tau(g)} g\right) \cdot \left(\sum_{m \in \mathbb{M}} b\right) = 1 \quad (2.1)$$

*The growth function of  $\mathbb{M}$  is equal to:*

$$F(t) = \sum_{m \in \mathbb{M}} t^{|m|s} = \left[ \sum_{g \in \mathcal{G}(\mathbb{M})} (-1)^{\tau(g)} t^{|g|s} \right]^{-1} \quad (2.2)$$

*Proof of Theorem 2.* For the passage from 2.1 to 2.2, the proof of Corollary 1 remains valid since we assume that  $\mathbb{M}$  is homogeneous.

For the first part of the theorem, we stay very close from the proof of Theorem 1 (see subsections 1.4 and 1.5 for more details). We construct an involution  $\Psi$  from  $(\mathcal{G}, \mathbb{M})$  to itself. We define the set  $\mathcal{E}$  of elements which will be allowed to move:

$$\mathcal{E} = \{\delta_J^{\{k\}}, \text{ where } J \subset \{s_1, \dots, s_n\} \text{ and } k \notin J\} \quad (2.3)$$

Definition 1 of an eligible moving part remains valid for a couple  $(g, m) \in \mathcal{G} \times \mathbb{M}$  (with the definition of  $\tau$  given in the end of section 2.1). The ordering

$s_1 < \dots < s_n$  on the generators induces a partial ordering on  $\mathcal{E}$  by the following:

$$\delta_J^{\{i\}} < \delta_{J'}^{\{i'\}} \text{ if and only if } i < i'$$

It is straightforward to see that given any couple of  $\mathcal{G} \times \mathbb{M} \setminus \{(1,1)\}$ , there is a unique maximal eligible moving part for this ordering, we call it the *moving part* and denote it  $k$ . We then define:

$$\Psi(g, m) = \begin{cases} (gk, k^{-1}m) & \text{if } k \prec m \\ (gk^{-1}, km) & \text{if } g \succ k \end{cases}$$

**Lemma 3.** The function  $\Psi$  is an involution from  $\mathcal{G} \times \mathbb{M}$  into itself whose unique fixed point is  $(1,1)$ .

If the lemma above is proved we can then conclude the proof of Theorem 2 exactly in the same way we did in subsection 1.5.  $\square$

*Proof of lemma 3.* The fact that  $(1,1)$  is the unique fixed point is clear (see proof of lemma 1 for details).

Let  $(g, m) \in \mathcal{G} \times \mathbb{M}$ , we set  $\Psi((g, m)) = (g', m')$  and we want to show that  $\Psi((g', m')) = (g, m)$ . As in the proof of lemma 1, we distinguish two cases: either the moving part was from  $g$  to  $m$  or from  $m$  to  $g$ .

We begin with the case of a moving part from  $m$  to  $g$ , we set:

$$\begin{cases} g = & \vee(J) \\ m = & \delta_J^{\{i\}} \cdot m_1, \end{cases} \quad \text{and} \quad \begin{cases} g' = & \vee(J) \cdot \delta_J^{\{i\}} = \vee(J \cup \{i\}) \\ m' = & m_1, \end{cases} \quad (2.4)$$

Assume the moving part for  $(g', m')$  is from  $m'$  to  $g'$ , it means that there exists  $l > i$  such that  $m' = \delta_{J \cup \{i\}}^{\{l\}} \cdot m'_1$ . Then:

$$g \cdot m = \vee(J) \cdot \delta_J^{\{i\}} \cdot \delta_{J \cup \{i\}}^{\{l\}} \cdot m'_1 \quad (2.5)$$

$$= \vee(J \cup \{i, l\}) \cdot m'_1 \quad (2.6)$$

$$= \vee(J) \cdot \delta_J^{\{l\}} \cdot \delta_{J \cup \{l\}}^{\{i\}} \cdot m'_1 \quad (2.7)$$

This contradicts the fact that  $\delta_J^{\{i\}}$  is the moving part from  $m$  to  $g$ . It remains to show that  $\forall j \in J, j < i$  which will imply that  $\delta_J^{\{i\}}$  is the maximal eligible moving part from  $g'$  to  $m'$ . Assume there exists  $j \in J$  such that  $j > i$ , then  $\delta_{J \setminus \{j\}}^{\{j\}}$  is an eligible moving part for  $(g, m)$  bigger than  $\delta_J^{\{i\}}$ , which contradicts the maximality of  $\delta_J^{\{i\}}$  and concludes the first case.

We deal now with the case of a moving part from  $g$  to  $m$  and set:

$$\begin{cases} g = & \vee(J \cup \{l\}) = \vee(J) \cdot \delta_J^{\{l\}} \\ m = & m, \end{cases} \quad \text{and} \quad \begin{cases} g' = & \vee(J) \\ m' = & \delta_J^{\{l\}} \cdot m \end{cases} \quad (2.8)$$

It is clear that the moving part for  $(g', m')$  is from  $m'$  to  $g'$  (otherwise it contradicts the maximality of  $l$  among elements of  $J \cup \{l\}$ ). Assume  $\delta_J^{\{l\}}$  is not the moving part for  $(g', m')$ , it implies that  $m' = \delta_J^{\{i\}} \cdot m_1$  with  $i > l$ . Hence we can write  $gm = g'm' = \vee(J \cup \{i\}) \cdot m_1$  which implies  $i \prec gm$ . Lastly for all  $j \in J$ ,  $j \prec \vee(J)$  and  $l \prec g$  hence :

$$\begin{aligned} \vee(J \cup \{l\} \cup \{i\}) &\prec gm \\ \vee(J \cup \{l\}) \cdot \delta_{J \cup \{l\}}^{\{i\}} &\prec gm \\ g \cdot \delta_{J \cup \{l\}}^{\{i\}} &\prec gm \end{aligned}$$

Now we assumed that  $\mathbb{M}$  has the left cancellation property thus  $\delta_{J \cup \{l\}}^{\{i\}} \prec m$  and  $\delta_{J \cup \{l\}}^{\{i\}}$  is an eligible moving part for  $(g, m)$  bigger than  $\delta_J^{\{l\}}$ . This contradicts the fact that  $\delta_J^{\{l\}}$  is the moving part for  $(g, m)$  and concludes the proof.  $\square$

## 2.3 A few examples of monoids

### 2.3.1 Artin-Tits monoids

Artin-Tits monoids are a generalisation of both braid and trace monoids. Given a finite set  $S$  and a symmetric matrix  $M = (m_{s,t})_{s,t \in S}$  such that  $m_{s,t} \in \mathbb{N} \cup \{\infty\}$  and  $m_{s,s} = 1$ , the *Artin-Tits monoid*  $\mathbb{M}$  associated to  $S$  and  $M$  has the following presentation:

$$\mathbb{M} = \langle s \in S \mid \underbrace{sts \dots}_{m_{s,t} \text{ terms}} = \underbrace{tst \dots}_{m_{s,t} \text{ terms}} \text{ if } m_{s,t} \neq \infty \rangle \quad (2.9)$$

Now an Artin-Tits monoid is clearly homogeneous, he has the left and right cancellation property (see Michel, Proposition 2.4 of [Mic99]) and has the least-common multiple property (see Brieskorn and Saito, Proposition 4.1 of [BS72])

### 2.3.2 Right-Gaussian monoids

In [DP99], Dehornoy and Paris generalise Artin-Tits groups. They define a *right Gaussian monoid* as a finitely generated monoid  $\mathbb{M}$  such that:

1. There exists a mapping  $\nu : \mathbb{M} \rightarrow \mathbb{N}$  such that  $\nu(a) < \nu(ab)$  for all  $a, b$  in  $\mathbb{M}$ ,  $b \neq 1$ .
2.  $\mathbb{M}$  is left cancellative.
3. All  $a, b \in \mathbb{M}$  admit a least common multiple.

Such a monoid is not necessarily homogeneous. We can nevertheless loosen conditions of Theorem 2 by only assuming that  $\mathbb{M}$  satisfies properties 2 and 3 of section 2.1 for a generating set  $\mathcal{S}$ . We define the set  $\mathcal{G}$  by choosing for each couple  $(a, b) \in \mathcal{S}^2$  one least common multiple. The first part of Theorem 2 remains true but it is no longer possible to compute the growth function of  $\mathbb{M}$ .

Now, if  $\mathbb{M}$  is a homogeneous right Gaussian monoid, we are exactly in the conditions of section 2.1. We apply this point to Birman-Ko-Lee braid monoids in the next section.

### 2.3.3 Birman-Ko-Lee braid monoids

In [BKL98], Birman, Ko and Lee gives a new presentation of braid groups. Namely the braid group on  $n$  strands has the following presentation:

$$\langle a_{ts}, n \geq t > s \geq 1 : a_{ts}a_{rq} = a_{rq}a_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0 \\ a_{ts}a_{sr} = a_{sr}a_{tr} = a_{tr}a_{ts} \text{ for } t > s > r \rangle$$

The new generators are conjugate of the classical Artin generators we use in the first section, indeed:

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\dots\sigma_{s+1})\sigma_s(\sigma_{t-1}\sigma_{t-2}\dots\sigma_{s+1})^{-1}$$

This new presentation gives birth to a new monoid denoted by  $B_n^{BKL+}$ . This monoid is a homogeneous right Gaussian monoid (and even a Garside monoid) hence it satisfies the conditions of section 2.1. We can then compute its growth function.

*Example* (Growth function of  $B_4^{BKL+}$ ). The generators of  $B_4^{BKL+}$  are  $a_{43}, a_{42}, a_{41}, a_{32}, a_{31}$  and  $a_{21}$  and they are submitted to relations:

$$\begin{aligned} a_{43}a_{32} &= a_{32}a_{42} = a_{42}a_{43} & a_{41}a_{32} &= a_{32}a_{41} \\ a_{43}a_{31} &= a_{31}a_{41} = a_{41}a_{43} & a_{43}a_{21} &= a_{21}a_{43} \\ a_{42}a_{21} &= a_{21}a_{41} = a_{41}a_{42} \\ a_{32}a_{21} &= a_{21}a_{31} = a_{31}a_{32} \end{aligned}$$

We observe that  $a_{43}a_{32}a_{21}$  is the least common multiple of the six generators and get:

$$F(t) = \frac{1}{1 - 6t + 10t^2 - 5t^3} \quad (2.10)$$

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